LOYOLA COLLEGE (AUTONOMOUS) CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

FOURTH SEMESTER – **APRIL 2025**

PMT4MC01 – FUNCTIONAL ANALYSIS

Date: 23-04-2025 Dept. No. Max. : 100 Marks

Time: 01:00 PM - 04:00 PM

	SECTION A – K1 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
1	**Answer the following**
a)	Define a normed linear space with an example.
b)	What is a Hilbert space?
c)	Define an orthonormal set in a Hilbert space.
d)	Explain self-adjoint operator on a Hilbert space.
e)	What do you mean by a normal operator on a Hilbert space?
	SECTION A – K2 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
2	**MCQ**
a)	A complete normed linear space is a ------------------------ (i) Metric space (ii) Banach space (iii) Topological space (iv) none of these.
b)	Which of the following is true? For $a \,\&\, b \geq 0 \; 1 \leq p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$ (i) $a^{\frac{1}{q}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$ (ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$ (iii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{b}{p} + \frac{a}{q}$ (iv) $a^{\frac{1}{q}} b^{\frac{1}{p}} \leq \frac{a}{p} + \frac{b}{q}$
c)	The set of all continuous linear transformations of a normed linear space *N* into another normed linear space N', $B(N, N')$ is a (i) Topological space (ii) Banach space (iii) Normed linear space (iv) none of the above
d)	An operator *N* on a Hilbert space *H* is unitary if--------- (i) $N^* = N$ (ii) $N\,N^* = N^*N$ (iii) $N\,N^* = N^*N = I$ (iv) none of the above
e)	If *N* is normal operator on a Hilbert space *H* then (i) $\|N^2\| = \|N\|$ (ii) $\|N^2\| = \|N^*\|$ (iii) $\|N^2\| = \|N\|^2$ (iv) none of the above.
	SECTION B – K3 (CO2)
	Answer any THREE of the following **(3 x 10 = 30)**
3	Prove Holder's inequality.
4	Is the set of all continuous linear transformations of a normed linear space *N* into another normed linear space N', $B(N, N')$ if N' is a Banch space a Banach space? Justify with a proof.
5	State and prove the four equivalent forms of continuous linear transformation *T* of a normed linear space *N* into another normed linear space N'.
6	Prove that if *M* is a proper closed linear subspace of a Hilbert space *H* there exists a non-zero vector z_0 in *H* such that $z_0 \perp M$.

7	Justify the following statement: "An operator *T* on a Hilbert space *H* is normal if and only if its real and imaginary parts commute".
	SECTION C – K4 (CO3)
	Answer any TWO of the following **(2 x 12.5 = 25)**
8	Examine the Hahn-Banach theorem.
9	Illustrate the uniform boundedness theorem and apply to prove "A non-empty subset *X* of a normed linear space *N* is bounded if and only if *f*(*X*) is a bounded set of numbers for each *f* in N^*"
10	Test the following: "If B is a complex Banach space whose norm obeys the parallelogram law and if an inner product is defined on B by $4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$ then *B* is a Hilbert space"
11	Analyze "if $\{e_1, e_2, e_3 \dots . e_n\}$ is a finite orthonormal set in a Hilbert space *H* and $x \in H$ then (i) $\sum_{i=1}^{n} \|(x, e_i)\|^2 \leq \|x\|^2$ (ii) $x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j$ for each j".
	SECTION D – K5 (CO4)
	Answer any ONE of the following **1 x 15 = 15)**
12	Assess the statement " An operator *T* on a Hilbert space *H* is unitary if and only if it is an isometric isomorphism of *H* onto itself"
13	Derive the set of all regular elements *G* of a Banach Algebra *A* is an open set and hence the set of all singular elements *S* of *A* is a closed set.
	SECTION E – K6 (CO5)
	Answer any ONE of the following **(1 x 20 = 20)**
14	Defend the following: (i) If N_1 and N_2 are normal operators on a Hilbert space *H* with the property that both commute with the adjoint of the other then $N_1 + N_2$ and $N_1 N_2$ are normal. (ii) An operator *T* on *H* is normal if and only if $\|T^*T\| = \|Tx\|$.
15	Derive the spectral radius of an element *x*, *r*(*x*), in a Banach Algebra *A* is $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$.

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